Mathematical Methods in Physics HW3

1. Consider the space P_2 of polynomials up to second order in t with the inner product defined by $(x, y) = \int_0^1 x(t)y(t)\, dt$. Using the standard basis $X = \{1, t, t^2\}$ as well as the orthonormal one we found in class $Y = \left\{1, \sqrt{12}\left(t - \frac{1}{2}\right), \sqrt{180}(t^2 - t + \frac{1}{6})\right\}$, evaluate the angle between the two vectors $x = 1 + 2t + 3t^2$ and $y = -2 + t^2$. Compare your results. Which method was easier?

We have that $\theta = \cos^{-1}\left(\frac{(x,y)}{\|x\|\|y\|}\right)$.

To use the standard basis, we need to remember cross terms and that the magnitude of basis vectors isn't one. So we have:

$$(x, y) = \int_0^1 (1 + 2t + 3t^2)(-2 + t^2)\, dt = \int_0^1 (-2 - 4t - 5t^2 + 2t^3 + 3t^4)\, dt = -\frac{137}{30}$$

$$\|x\| = \sqrt{\int_0^1 (1 + 2t + 3t^2)(1 + 2t + 3t^2) dt} = \sqrt{\int_0^1 (1 + 4t + 10t^2 + 12t^3 + 9t^4) dt} = \sqrt{\frac{167}{15}}$$

$$\|y\| = \sqrt{\int_0^1 (-2 + t^2)(-2 + t^2) dt} = \sqrt{\int_0^1 (4 - 4t^2 + t^4) dt} = \sqrt{\frac{43}{15}}$$

Putting it all together we find $\theta = 143.9^o$.

To use the orthonormal basis we first have to find the vectors in that basis. After that, everything should be much easier. So for any vector v starting in the X basis, we can rewrite it in the Y basis using:

$$v = \alpha_0 + \alpha_1\, 2\sqrt{3}\left(t - \frac{1}{2}\right) + \alpha_2\, 6\sqrt{5}\left(t^2 - t + \frac{1}{6}\right) = (\alpha_0 - \alpha_1\sqrt{3} + \alpha_2\sqrt{5}) + (\alpha_1 2\sqrt{3} - \alpha_2 6\sqrt{5})t + \alpha_2 t^2$$

Equating terms to $x = 1 + 2t + 3t^2$ we find $\alpha_2 = \frac{1}{2\sqrt{5}}, \alpha_1 = \frac{5}{2\sqrt{3}}, \alpha_0 = 3$, so $x = 3y_0 + \frac{5}{2\sqrt{3}}y_1 + \frac{1}{2\sqrt{5}}y_2$.

Similarly, for $y = -2 + t^2$ we find $\alpha_2 = \frac{1}{6\sqrt{5}}, \alpha_1 = \frac{1}{2\sqrt{3}}, \alpha_0 = -\frac{5}{3}$, so $y = -\frac{5}{3}y_0 + \frac{1}{2\sqrt{3}}y_1 + \frac{1}{6\sqrt{5}}y_2$.

Now we can calculate θ by largely ignoring the orthonormal basis and just focusing on the coeffiecients:

$$(x, y) = 3\left(-\frac{5}{3}\right) + \frac{5}{2\sqrt{3}}\frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{5}}\frac{1}{6\sqrt{5}} = -\frac{137}{30}$$

$$\|x\| = \sqrt{3 \cdot 3 + \frac{5}{2\sqrt{3}}\frac{5}{2\sqrt{3}} + \frac{1}{2\sqrt{5}}\frac{1}{2\sqrt{5}}} = \sqrt{\frac{167}{15}}$$

$$\|y\| = \sqrt{-\frac{5}{3} \cdot -\frac{5}{3} + \frac{1}{2\sqrt{3}}\frac{1}{2\sqrt{3}} + \frac{1}{6\sqrt{5}}\frac{1}{6\sqrt{5}}} = \sqrt{\frac{43}{15}}$$

Thus we get the same angle as when using the other basis. I would say that starting with the original basis was a bit quicker since that is the readily identifiable basis in the original polynomial expressions.

2. Consider the vector space of even polynomials up to 4th degree armed with the inner product defined by $(x,y) = \int_0^1 x(t)y(t)\, dt$. A natural basis would be $X = \{x_0, x_2, x_4\} = \{1, t^2, t^4\}$.

a) Confirm that this basis is not orthogonal.

$$(1, t^2) = \int_0^1 t^2\, dt = \frac{1}{3} \neq 0\ ,\ (1, t^4) = \int_0^1 t^4\, dt = \frac{1}{5} \neq 0\ ,\ (t^2, t^4) = \int_0^1 t^6\, dt = \frac{1}{7} \neq 0$$

b) Starting with the basis vector x_0, use the Graham-Schmidt process to determine an orthonormal basis.

$$y_1 = \frac{x_0}{\|x_0\|} = 1$$

$$y_2 = \frac{x_2 - (y_1, x_2)y_1}{\|x_2 - (y_1, x_2)y_1\|} = \frac{t^2 - \int_0^1 t^2\, dt}{\left\|t^2 - \int_0^1 t^2\, dt\right\|} = \frac{t^2 - \frac{1}{3}}{\sqrt{\int_0^1 (t^2 - \frac{1}{3})^2\, dt}} = \frac{\sqrt{45}}{2}\left(t^2 - \frac{1}{3}\right)$$

$$y_3 = \frac{x_4 - (y_1, x_4)y_1 - (y_2, x_4)y_2}{\|x_4 - (y_1, x_4)y_1 - (y_2, x_4)y_2\|} = \frac{t^4 - \int_0^1 t^4\, dt - \frac{\sqrt{45}}{2}(t^2 - \frac{1}{3})\int_0^1 \frac{\sqrt{45}}{2}(t^2 - \frac{1}{3})t^4\, dt}{\left\|t^4 - \int_0^1 t^4\, dt - \frac{\sqrt{45}}{2}(t^2 - \frac{1}{3})\int_0^1 \frac{\sqrt{45}}{2}(t^2 - \frac{1}{3})t^4\, dt\right\|}$$

$$= \frac{t^4 - \frac{1}{5} - \frac{45}{4}(t^2 - \frac{1}{3})\frac{8}{105}}{\left\|t^4 - \frac{1}{5} - \frac{45}{4}(t^2 - \frac{1}{3})\frac{8}{105}\right\|} = \frac{t^4 - \frac{6}{7}t^2 + \frac{3}{35}}{\left\|t^4 - \frac{6}{7}t^2 + \frac{3}{35}\right\|} = \sqrt{\frac{64}{11025}}\left(t^4 - \frac{6}{7}t^2 + \frac{3}{35}\right) = \frac{8}{105}\left(t^4 - \frac{6}{7}t^2 + \frac{3}{35}\right)$$

c) Starting this time with the basis vector x_2, use Graham-Schmidt to determine an orthonormal basis.

$$y_1 = \frac{x_2}{\|x_2\|} = \frac{t^2}{\sqrt{\int_0^1 t^4 dt}} = \sqrt{5}t^2$$

$$y_2 = \frac{x_4 - (y_1, x_4)y_1}{\|x_4 - (y_1, x_4)y_1\|} = \frac{t^4 - \sqrt{5}t^2 \int_0^1 \sqrt{5}t^6\, dt}{\left\|t^4 - \sqrt{5}t^2 \int_0^1 \sqrt{5}t^6\, dt\right\|} = \frac{t^4 - \frac{5}{7}t^2}{\sqrt{\int_0^1 (t^4 - \frac{5}{7}t^2)^2\, dt}} = \frac{2}{21}\left(t^4 - \frac{5}{7}t^2\right)$$

$$y_3 = \frac{x_0 - (y_1, x_0)y_1 - (y_2, x_0)y_2}{\|x_0 - (y_1, x_0)y_1 - (y_2, x_0)y_2\|} = \frac{1 - \frac{5}{3}t^2 + \frac{16}{46305}(t^4 - \frac{5}{7}t^2)}{\sqrt{\int_0^1 \left[1 - \frac{5}{3}t^2 + \frac{16}{46305}(t^4 - \frac{5}{7}t^2)\right]^2 dt}} = N\left[1 - \frac{5}{3}t^2 + \frac{16}{46305}(t^4 - \frac{5}{7}t^2)\right]$$

3. Consider the derivative operators $D = \frac{d}{dt}$ and $D^2 = \frac{d^2}{dt^2}$ acting on the vector space considered in question (2).

a) Are both of these linear operators on this space? Explain.

The first derivative takes the basis from $\{1, t^2, t^4\}$ to $\{0, 2t, 4t^3\}$. The last two are not elements of the original vector space, so D is not a linear operator.

The second derivative takes the basis from $\{1, t^2, t^4\}$ to $\{0, 2, 12t^2\}$. All of these are elements of the original vector space, so D^2 is a linear operator.

b) For any of these two that are linear operators, find their matrix representations with respect to the three bases discussed in problem (2). Verify the matrix forms by acting on the vector $x = \alpha_0 + \alpha_2 t^2 + \alpha_4 t^4$ (rewritten in terms of each basis) with both the straight derivative(s) and via matrix multiplication of the components.

First of all, $D^2 x = 2\alpha_2 + 12\alpha_4 t^2$ for the first basis so $D^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{pmatrix}$.

For the second basis we have $x = \left(\alpha_0 + \frac{\alpha_2}{3} + \frac{\alpha_4}{5}\right)y_1 + \left(\frac{2}{\sqrt{45}}\alpha_2 - \frac{12}{7\sqrt{45}}\alpha_4\right)y_2 + \frac{105}{8}\alpha_4 y_3$ and

then $D^2 x = (2\alpha_2 + 4\alpha_4)y_1 + \frac{24}{\sqrt{45}}\alpha_4 y_2$. Then we need a matrix D^2 s.t.

$$D^2 \begin{pmatrix} \alpha_0 + \frac{\alpha_2}{3} + \frac{\alpha_4}{5} \\ \frac{2}{\sqrt{45}}\alpha_2 - \frac{12}{7\sqrt{45}}\alpha_4 \\ \frac{105}{8}\alpha_4 \end{pmatrix} = \begin{pmatrix} 2\alpha_2 + 4\alpha_4 \\ \frac{24}{\sqrt{45}}\alpha_4 \\ 0 \end{pmatrix} \Rightarrow D^2 = \begin{pmatrix} 0 & \sqrt{45} & \frac{320}{735} \\ 0 & 0 & \frac{192}{105\sqrt{45}} \\ 0 & 0 & 0 \end{pmatrix}.$$

c) Find the transformation that carries the operators between each of the three basis sets.

To find the transformation we can start by transforming the vector components of x in the first basis to the components in the second basis:

$$U \begin{pmatrix} \alpha_0 \\ \alpha_2 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 + \frac{\alpha_2}{3} + \frac{\alpha_4}{5} \\ \frac{2}{\sqrt{45}}\alpha_2 - \frac{12}{7\sqrt{45}}\alpha_4 \\ \frac{105}{8}\alpha_4 \end{pmatrix} \Rightarrow U = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{5} \\ 0 & \frac{2}{3\sqrt{5}} & -\frac{4}{7\sqrt{5}} \\ 0 & 0 & \frac{105}{8} \end{pmatrix}$$

Then we find its inverse:

$$U^{-1} = \begin{pmatrix} 1 & -\frac{\sqrt{5}}{2} & -\frac{136}{3675} \\ 0 & \frac{3\sqrt{5}}{2} & \frac{48}{735} \\ 0 & 0 & \frac{8}{105} \end{pmatrix}$$

And now we transform:

$$D^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{pmatrix} \to UD^2U^{-1} = \begin{pmatrix} 0 & \sqrt{45} & \frac{320}{735} \\ 0 & 0 & \frac{192}{105\sqrt{45}} \\ 0 & 0 & 0 \end{pmatrix}.$$